On the Symmetry of the Permeability Tensor

Before considering the permeability, let us examine the energy dissipation associated with Darcy flow. We use the physical principle that it is unrealistic for a flow to exist without energy dissipation. (At the microscopic scale, any fluid motion necessarily involves frictional pressure losses.)

Let $\boldsymbol{\sigma}$ denote the stress tensor. The Gauss divergence theorem states that

$$\int_{\Omega} \nabla \cdot \boldsymbol{\sigma} \, dV = \int_{\partial\Omega} \boldsymbol{\sigma} \cdot d\boldsymbol{S}$$

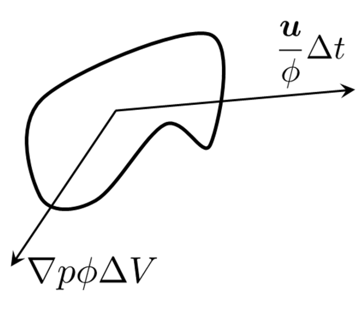
where $\partial\Omega$ is the closed surface and the domain inside $\partial\Omega$ is Ω.

If we consider only the fluid pressure p, this reduces to

$$\int_{\Omega} \nabla p \, dV = \int_{\partial\Omega} p \, d\boldsymbol{S}$$

Hence, the force acting on an infinitesimal volume of fluid is given by the product of the pressure gradient and the fluid volume. For a fluid element of volume ΔV over a time interval Δt, the work done by the external force is

$$(\nabla p \phi \Delta V) \cdot \left(\frac{\boldsymbol{u}}{\phi} \Delta t \right) = (\nabla p) \cdot \boldsymbol{u} \Delta V \Delta t$$



This implies that the energy dissipation per unit volume and per unit time is given by

$$-(\nabla p) \cdot \boldsymbol{u}$$

Darcy's law is given by

$$\boldsymbol{u} = -\frac{\boldsymbol{k}}{\mu} \nabla p$$

where $\boldsymbol{k}$ is the permeability tensor and μ is the fluid viscosity.

Here, we assume that the permeability tensor is not symmetric. We decompose the flux as

$$\boldsymbol{u} = \boldsymbol{u}^s + \boldsymbol{u}^a = -\frac{\boldsymbol{k}^s}{\mu}\nabla p - \frac{\boldsymbol{k}^a}{\mu}\nabla p$$

where $\boldsymbol{k} = \boldsymbol{k}^s + \boldsymbol{k}^a$, with

$$k_{ij}^s = \frac{k_{ij} + k_{ji}}{2}$$

$$k_{ij}^a = \frac{k_{ij} - k_{ji}}{2}$$

Accordingly,

$$\boldsymbol{u}^s = -\frac{\boldsymbol{k}^s}{\mu}\nabla p$$

$$\boldsymbol{u}^a = -\frac{\boldsymbol{k}^a}{\mu}\nabla p$$

By construction, $k_{ij}^s = k_{ji}^s$ and $k_{ij}^a = -k_{ji}^a$. Since we assume that the permeability tensor is not symmetric, there exists a pressure gradient ∇p such that $\boldsymbol{u}^a \neq 0$.

The energy dissipation at a point is therefore

$$-(\nabla p) \cdot \boldsymbol{u} = \frac{1}{\mu}(\nabla p) \cdot \boldsymbol{k}^s \cdot (\nabla p) + \frac{1}{\mu}(\nabla p) \cdot \boldsymbol{k}^a \cdot (\nabla p)$$

The second term on the right-hand side is

$$\frac{1}{\mu}(\nabla p) \cdot \boldsymbol{k}^a \cdot (\nabla p) = \sum_{i,j} \frac{\partial p}{\partial x_i} k_{ij}^a \frac{\partial p}{\partial x_j}$$

Since $k_{ij}^a = -k_{ji}^a$,

$$\begin{aligned}\frac{1}{\mu}(\nabla p) \cdot \boldsymbol{k}^a \cdot (\nabla p) &= -\sum_{i,j} \frac{\partial p}{\partial x_i} k_{ji}^a \frac{\partial p}{\partial x_j} \\ &= -\sum_{i,j} \frac{\partial p}{\partial x_j} k_{ij}^a \frac{\partial p}{\partial x_i}\end{aligned}$$

This implies

$$\frac{1}{\mu}(\nabla p) \cdot \boldsymbol{k}^a \cdot (\nabla p) = 0$$

Therefore, if the permeability tensor is not symmetric, there exists a pressure gradient ∇p such that $\boldsymbol{u}^a \neq 0$ while no energy dissipation occurs. This contradicts the fundamental physical requirement that any flow must be accompanied by energy dissipation. Hence, the permeability tensor must be symmetric.

We proved the symmetry of the permeability tensor based on energy dissipation arguments. Alternatively, the symmetry of the permeability tensor is commonly derived using Onsager's reciprocal relations. Although I have not yet fully understood this approach, I intend to study it further and include such a derivation in a future version of this document.