Gaussian Integral

Sometimes, we need to evaluate the following integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Surprisingly, this is equal to $\sqrt{\pi}$.

There are several ways to evaluate this integral, but an elegant and simple method is shown here.

Let I be the value of the integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

We consider I^2:

$$\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 \\
&= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-x^2}\,dx \\
&= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy
\end{aligned}$$

Then, Next, we change variables from Cartesian coordinates (x, y) to polar coordinates (r, θ):

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\,dr\,d\theta \\
&= 2\pi \int_{0}^{\infty} re^{-r^2}\,dr \\
&= 2\pi \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \\
&= \pi
\end{aligned}$$

Since the integrand is always positive ($e^{-x^2} > 0$), we conclude that $I > 0$. Therefore,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = I = \sqrt{\pi}$$