Why Radians

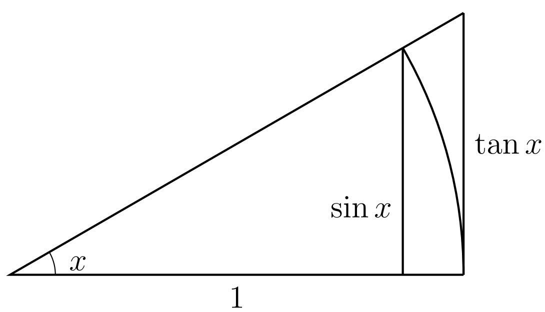
When I first learned about radians in high school, I wondered why angles are defined in such a strange way.

At that time, I was taught that the arc length can be expressed as $r\theta$ (where r is the radius and θ is the angle in radians). I didn't think this was very meaningful.

However, when I learned about differentiation, I finally understood the purpose of introducing radians. What I will explain below may be something many people are already familiar with, but I believe it is a point that deserves more emphasis when we teach what radians are really for.

Let x be a positive angle in degrees. (The same discussion applies if x is negative.)

Now, consider the following arc and the two triangles shown below:



The arc length lies between the heights of the two triangles:

$$\sin x \leq 2\pi \frac{x}{360} \leq \tan x$$

Dividing all sides by $\sin x$, we obtain:

$$\frac{180}{\pi} \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \frac{180}{\pi}$$

Taking the reciprocal:

$$\cos x \frac{\pi}{180} \leq \frac{\sin x}{x} \leq \frac{\pi}{180}$$

Therefore,

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\pi}{180}$$

Now, consider the derivative of $\sin x$:

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} &= \lim_{\Delta x \to 0} \frac{2\cos\frac{2x+\Delta x}{2}\sin\frac{\Delta x}{2}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \\ &= \frac{\pi}{180}\cos x\end{aligned}$$

As you can see, we would need to multiply by $\dfrac{\pi}{180}$ every time we take a derivative. This is quite cumbersome.

But if we use radians instead:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

So we no longer need to multiply by $\dfrac{\pi}{180}$.

This is a more convincing reason to introduce radians than simply saying they express arc length.