Somewhere on Earth, Antipodal Points Must Share a Temperature

There must be a pair of opposite points on Earth that are equally warm. This fact can be proven under a simple assumption: that temperature varies continuously over the Earth's surface.

Let temperature be a function of position, denoted $f : S^2 \to \mathbb{R}$, where S^2 represents the surface of a sphere. Now, consider the function

$$g(\vec{x}) = f(\vec{x}) - f(-\vec{x})$$

This function represents the temperature difference between a point and its antipode. Clearly, g is an odd function, meaning $g(\vec{x}) = -g(-\vec{x})$.

Suppose that for some point $\vec{x_0}$, we have $g(\vec{x_0}) > 0$. (If $g(\vec{x_0}) < 0$, we simply let $\vec{x_0}$ be $-\vec{x_0}$ instead. If $g(\vec{x_0}) = 0$, then $f(\vec{x}) = f(-\vec{x})$, meaning the temperatures already match!) Then, by the oddness of g, we have $g(-\vec{x_0}) = -g(\vec{x_0}) < 0$ Now, consider any continuous path on the sphere connecting $\vec{x_0}$ and $-\vec{x_0}$.

By the Intermediate Value Theorem, there must exist a point along this path where $g(\vec{x}) = 0$. That is, $f(\vec{x}) = f(-\vec{x})$: the temperature at some point on Earth must be equal to that at its antipode.

This result is a special case of the Borsuk–Ulam Theorem, which more generally states that any continuous function from an n-sphere to $\mathbb{R}^n$ maps some pair of antipodal points to the same value.

In fact, using this theorem, we can show that there exists a pair of antipodal points on Earth with not only the same temperature, but also the same pressure. Maybe I'll write about that another time.