Inverse Fourier Transform Calculation Example Using Cauchy's Integral Formula

Although there are several conventions for the Fourier Transform, the following definition is used here.

$$\mathcal{F}\left[f\right](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$$

The corresponding inverse Fourier Transform is given by

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi$$

As an example, we consider the following function:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + \xi^2}$$

The Inverse Fourier Transform is

$$\begin{aligned}
\mathcal{F}^{-1}\left[\hat{f}\right](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + \xi^2} e^{i\xi x}\, d\xi \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{1}{\xi - ia} - \frac{1}{\xi + ia}\right] e^{i\xi x}\, d\xi \\
&= \frac{1}{2\pi i} \left[\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi - ia}\, d\xi - \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi + ia}\, d\xi\right]
\end{aligned}$$

We can evaluate the integral using Cauchy's integral formula, which is

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

where f is an analytic function within the domain closed by C. In addition, z_0 is in the domain closed by C. ($e^{i\xi x}$ is analytic at any point on the complex plane.)

If $x > 0$, we consider the semi-disk shown in Figure 1.

Since there are no singularities for $\dfrac{e^{izx}}{z + ia}$ inside the contour, we can apply Cauchy's Integral Theorem:

$$\frac{1}{2\pi i} \oint_C \frac{e^{izx}}{z + ia}\, dz = 0$$

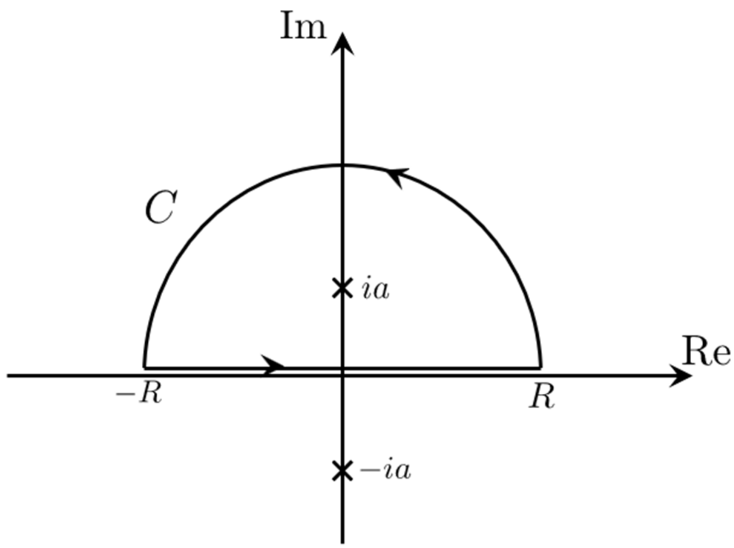

Figure 1: Contour C for integration

Since e^{izx} does not have singularity inside the contour, we can apply Cauchy's integral formula:

$$\frac{1}{2\pi i}\oint_C \frac{e^{izx}}{z-ia}\,dz = e^{-ax}$$

Additionally, the above contour integrals can be written as

$$\frac{1}{2\pi i}\oint_C \frac{e^{izx}}{z+ia}\,dz = \frac{1}{2\pi i}\int_{-R}^{R} \frac{e^{i\xi x}}{\xi+ia}\,d\xi + \frac{1}{2\pi i}\int_0^{\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta}+ia} iRe^{i\theta}\,d\theta = 0$$

$$\frac{1}{2\pi i}\oint_C \frac{e^{izx}}{z-ia}\,dz = \frac{1}{2\pi i}\int_{-R}^{R} \frac{e^{i\xi x}}{\xi-ia}\,d\xi + \frac{1}{2\pi i}\int_0^{\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta}-ia} iRe^{i\theta}\,d\theta = e^{-ax}$$

Here, we evaluate the following integral

$$\int_0^{\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta}\pm ia} iRe^{i\theta}\,d\theta$$

$$
\begin{aligned}
\left|\int_0^{\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta}\, d\theta\right| &\leq \int_0^{\pi} \left|\frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta}\right| d\theta \\
&= \int_0^{\pi} \frac{\left|e^{iRe^{i\theta}x}\right|}{|Re^{i\theta} \pm ia|} R\, d\theta \\
&= \int_0^{\pi} \frac{\left|e^{iR(\cos\theta + i\sin\theta)x}\right|}{|Re^{i\theta} \pm ia|} R\, d\theta \\
&= \int_0^{\pi} \frac{e^{-R\sin\theta x}}{|Re^{i\theta} \pm ia|} R\, d\theta \\
&\leq \int_0^{\pi} \frac{e^{-R\sin\theta x}}{|R-a|} R\, d\theta \quad (\text{Using } ||z_1| - |z_2|| \leq |z_1 - z_2|) \\
&= \int_0^{\pi} \frac{e^{-R\sin\theta x}}{R-a} R\, d\theta \quad (R > a \text{ since } ia \text{ is in the semi-disk.}) \\
&= \frac{R}{R-a} \int_0^{\pi} e^{-R\sin\theta x}\, d\theta \\
&= \frac{R}{R-a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-R\sin\left(\theta + \frac{\pi}{2}\right)x}\, d\theta \\
&= \frac{R}{R-a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-R\cos\theta x}\, d\theta \\
&= \frac{2R}{R-a} \int_0^{\frac{\pi}{2}} e^{-R\cos\theta x}\, d\theta \quad (\cos\theta \text{ is an even function})
\end{aligned}
$$

For the interval of $0 \leq \theta \leq \frac{\pi}{2}$, we can graphically confirm that $\cos\theta \geq -\frac{2}{\pi}\theta + 1$. Thus,

$$
\begin{aligned}
\left|\int_0^{\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta}\, d\theta\right| &\leq \frac{2R}{R-a} \int_0^{\frac{\pi}{2}} e^{-R\left(-\frac{2}{\pi}\theta + 1\right)x}\, d\theta \\
&= \frac{2Re^{-Rx}}{R-a} \int_0^{\frac{\pi}{2}} e^{R\frac{2}{\pi}\theta x}\, d\theta \\
&= \frac{2Re^{-Rx}}{R-a} \left[\frac{\pi}{2Rx} e^{R\frac{2}{\pi}\theta x}\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi e^{-Rx}}{(R-a)\, x} \left(e^{Rx} - 1\right) \\
&= \frac{\pi\left(1 - e^{-Rx}\right)}{(R-a)\, x} \to 0 \quad (R \to \infty)
\end{aligned}
$$

Therefore,

$$
\lim_{R\to\infty} \left[\frac{1}{2\pi i} \int_{-R}^{R} \frac{e^{i\xi x}}{\xi + ia}\, d\xi + \frac{1}{2\pi i} \int_0^{\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} + ia} iRe^{i\theta}\, d\theta\right] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi + ia}\, d\xi = 0
$$

$$\lim_{R\to\infty}\left[\frac{1}{2\pi i}\int_{-R}^{R}\frac{e^{i\xi x}}{\xi - ia}\,d\xi + \frac{1}{2\pi i}\int_{0}^{\pi}\frac{e^{iRe^{i\theta}x}}{Re^{i\theta} - ia} iRe^{i\theta}\,d\theta\right] = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi - ia}\,d\xi = e^{-ax}$$

Hence, the Inverse Fourier Transform for $x > 0$ is

$$\begin{aligned}\mathcal{F}^{-1}\left[\hat{f}\right](x) &= \frac{1}{2\pi i}\left[\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi - ia}\,d\xi - \int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi + ia}\,d\xi\right] \\ &= e^{-ax}\end{aligned}$$

If $x < 0$, we consider the semi-disk shown in Figure 2.

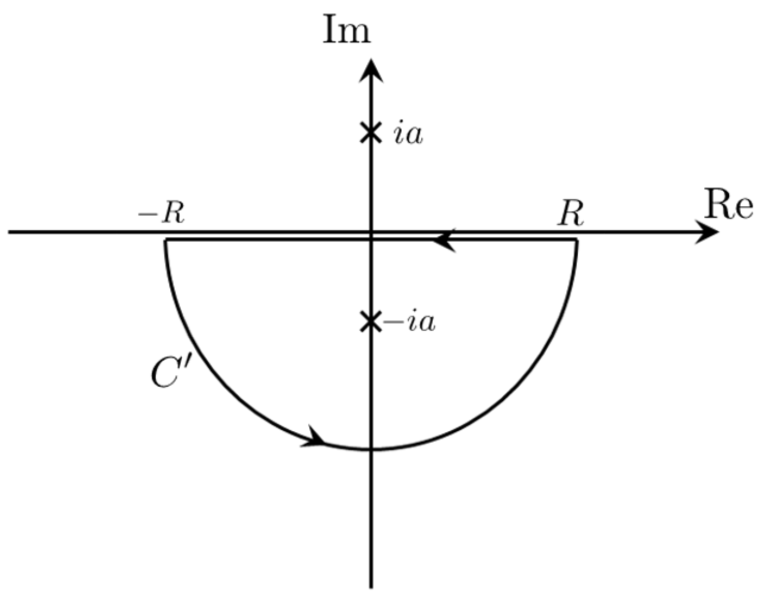

Figure 2: Contour C' for integration

We repeat the same calculation using the different contour. Since there are no singularities for $\dfrac{e^{izx}}{z - ia}$ inside the contour, we can apply Cauchy's Integral Theorem:

$$\frac{1}{2\pi i}\oint_{C'}\frac{e^{izx}}{z - ia}\,dz = 0$$

Since e^{izx} does not have singularity inside the contour, we can apply Cauchy's integral formula:

$$\frac{1}{2\pi i}\oint_{C'}\frac{e^{izx}}{z + ia}\,dz = e^{ax}$$

Additionally, the above contour integrals can be written as

$$\frac{1}{2\pi i}\oint_{C'}\frac{e^{izx}}{z - ia}\,dz = \frac{1}{2\pi i}\int_{R}^{-R}\frac{e^{i\xi x}}{\xi - ia}\,d\xi + \frac{1}{2\pi i}\int_{\pi}^{2\pi}\frac{e^{iRe^{i\theta}x}}{Re^{i\theta} - ia} iRe^{i\theta}\,d\theta = 0$$

$$\frac{1}{2\pi i}\oint_{C'}\frac{e^{izx}}{z+ia}\,dz=\frac{1}{2\pi i}\int_{R}^{-R}\frac{e^{i\xi x}}{\xi+ia}\,d\xi+\frac{1}{2\pi i}\int_{\pi}^{2\pi}\frac{e^{iRe^{i\theta}x}}{Re^{i\theta}+ia}iRe^{i\theta}\,d\theta=e^{ax}$$

Here, we evaluate the following integral

$$\int_{\pi}^{2\pi}\frac{e^{iRe^{i\theta}x}}{Re^{i\theta}\pm ia}iRe^{i\theta}\,d\theta$$

$$\begin{aligned}\left|\int_{\pi}^{2\pi}\frac{e^{iRe^{i\theta}x}}{Re^{i\theta}\pm ia}iRe^{i\theta}\,d\theta\right| &\leq \frac{R}{R-a}\int_{\pi}^{2\pi}e^{-R\sin\theta x}\,d\theta \quad \text{(The same calculation as } x>0\text{)}\\ &=\frac{R}{R-a}\int_{0}^{\pi}e^{R\sin\theta x}\,d\theta\\ &=\frac{R}{R-a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{R\sin\left(\theta+\frac{\pi}{2}\right)x}\,d\theta\\ &=\frac{R}{R-a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{R\cos\theta x}\,d\theta\\ &=\frac{2R}{R-a}\int_{0}^{\frac{\pi}{2}}e^{R\cos\theta x}\,d\theta \quad (\cos\theta \text{ is an even function})\end{aligned}$$

Since $\cos\theta\geq-\frac{2}{\pi}\theta+1\ \left(0\leq\theta\leq\frac{\pi}{2}\right)$ and $x<0$,

$$\begin{aligned}\left|\int_{\pi}^{2\pi}\frac{e^{iRe^{i\theta}x}}{Re^{i\theta}\pm ia}iRe^{i\theta}\,d\theta\right| &\leq \frac{2R}{R-a}\int_{0}^{\frac{\pi}{2}}e^{R\left(-\frac{2}{\pi}\theta+1\right)x}\,d\theta\\ &=\frac{2Re^{Rx}}{R-a}\int_{0}^{\frac{\pi}{2}}e^{-R\frac{2}{\pi}\theta x}\,d\theta\\ &=\frac{2Re^{Rx}}{R-a}\left[\frac{\pi}{2Rx}e^{-R\frac{2}{\pi}\theta x}\right]_{0}^{\frac{\pi}{2}}\\ &=\frac{\pi e^{Rx}}{(R-a)\,x}\left(e^{-Rx}-1\right)\\ &=\frac{\pi\left(1-e^{Rx}\right)}{(R-a)\,x}\to 0 \quad (R\to\infty)\end{aligned}$$

Therefore,

$$\lim_{R\to\infty}\left[\frac{-1}{2\pi i}\int_{-R}^{R}\frac{e^{i\xi x}}{\xi-ia}\,d\xi+\frac{1}{2\pi i}\int_{\pi}^{2\pi}\frac{e^{iRe^{i\theta}x}}{Re^{i\theta}-ia}iRe^{i\theta}\,d\theta\right]=\frac{-1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi-ia}\,d\xi=0$$

$$\lim_{R\to\infty}\left[\frac{-1}{2\pi i}\int_{-R}^{R}\frac{e^{i\xi x}}{\xi+ia}\,d\xi+\frac{1}{2\pi i}\int_{\pi}^{2\pi}\frac{e^{iRe^{i\theta}x}}{Re^{i\theta}+ia}iRe^{i\theta}\,d\theta\right]=\frac{-1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi+ia}\,d\xi=e^{ax}$$

Hence, the Inverse Fourier Transform for $x < 0$ is

$$\begin{aligned}\mathcal{F}^{-1}\left[\hat{f}\right](x) &= \frac{1}{2\pi i}\left[\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi - ia}\,d\xi - \int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi + ia}\,d\xi\right] \\ &= e^{ax}\end{aligned}$$

If $x = 0$, The Inverse Fourier Transform is

$$\begin{aligned}\mathcal{F}^{-1}\left[\hat{f}\right](0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+\xi^2}\,d\xi \\ &= \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{1}{a^2+\xi^2}\,d\xi \\ &= \frac{a}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{a^2+a^2\tan^2\phi}\frac{a}{\cos^2\phi}\,d\phi \quad (\xi = a\tan\phi) \\ &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\,d\phi \\ &= 1\end{aligned}$$

Therefore,

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) = \begin{cases} e^{-ax} & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ e^{ax} & \text{if } x < 0 \end{cases}$$

This is equivalent to

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) = e^{-a|x|}$$