Divergence and Curl

There are several forms of the difinition of divergence and curl. One of them is using the integral form.

Definition 1 Divergence. *The divergence of a vector field $\vec{v}$ is defined as*

$$\nabla \cdot \vec{v} = \lim_{V \to 0} \frac{\displaystyle\int_{\partial\Omega} \vec{v} \cdot \vec{dS}}{V}$$

where $\partial\Omega$ is the closed surface and the volume inside $\partial\Omega$ is V

Definition 2 Curl. *The curl of a vector field $\vec{v}$ is defined as*

$$(\nabla \times \vec{v}) \cdot \vec{n} = \lim_{S \to 0} \frac{\displaystyle\oint_C \vec{v} \cdot \vec{dr}}{S}$$

where $\vec{n}$ is a unit vector in an arbitrary direction, S is an area of plane perpendicular to $\vec{n}$ and closed by curve C

Theorem Gauss Divergence Theorem.

$$\int_{\Omega} \nabla \cdot \vec{v} dV = \int_{\partial\Omega} \vec{v} \cdot \vec{dS}$$

Proof. First, we divide the region Ω into n regions. Each region is denoted as $\Omega_i(i = 1, 2, \cdots, n)$. We can write the RHS of the theorem as follows.

$$\int_{\partial\Omega} \vec{v} \cdot \vec{dS} = \sum_{i=1}^{n} \int_{\partial\Omega_i} \vec{v} \cdot \vec{dS}$$

This is because the surface integral on the boundary between Ω_i and Ω_j cancels out due to the oposite direction of $\vec{dS}$. Let V_i be the volume of the region Ω_i. In addition, we define $|\Delta| = \max\{V_i; 1 \leq i \leq n\}$. We can increase the number of division so that $|\Delta|$ becomes less than any positive value. Therefore,

$$\begin{aligned}\int_{\partial\Omega} \vec{v}\cdot d\vec{S} &= \sum_{i=1}^{n} \int_{\partial\Omega_i} \vec{v}\cdot d\vec{S} \\ &= \sum_{i=1}^{n} \frac{\int_{\partial\Omega_i} \vec{v}\cdot d\vec{S}}{V_i} V_i \\ &\to \int_{\Omega} \nabla\cdot\vec{v} dV \quad (|\Delta| \to 0)\end{aligned}$$

□

Theorem Stokes' Theorem.

$$\int_{\Gamma} (\nabla\times\vec{v})\cdot d\vec{s} = \oint_C \vec{v}\cdot d\vec{r}$$

Proof. First, we divide the surface Γ into n surfaces. Each surface is denoted as $\Gamma_i(i = 1, 2, \cdots, n)$, and the closed loop around Γ_i is C_i We can write the RHS of the theorem as follows.

$$\oint_C \vec{v}\cdot d\vec{r} = \sum_{i=1}^{n} \oint_{C_i} \vec{v}\cdot d\vec{r}$$

This is because the integral on the boundary between Γ_i and Γ_j cancels out due to the opposite direction of the integral. Let S_i be the are of the surface Γ_i. In addition, we define $|\Delta| = \max\{S_i; 1 \le i \le n\}$. We can increase the number of division so that $|\Delta|$ becomes less than any positive value. Therefore,

$$\begin{aligned}\oint_C \vec{v}\cdot d\vec{r} &= \sum_{i=1}^{n} \oint_{C_i} \vec{v}\cdot d\vec{r} \\ &= \sum_{i=1}^{n} \frac{\oint_{C_i} \vec{v}\cdot d\vec{r}}{S_i} S_i \\ &\to \int_{\Gamma} (\nabla\times\vec{v})\cdot d\vec{S} \quad (|\Delta| \to 0)\end{aligned}$$

□

We can derive the divergence in Cartesian coordinate by considering the rectangular prism shown in Figure 1.

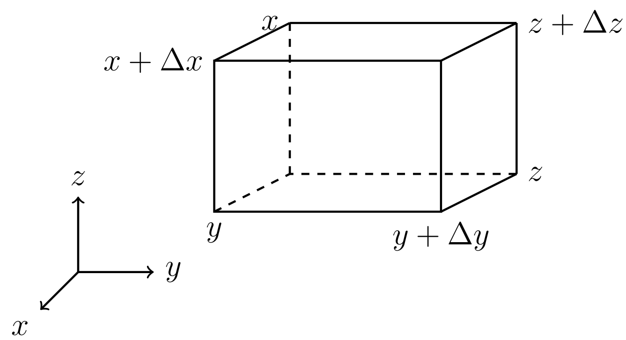

Figure 1: Rectangular prism

Using the Gauss divergence theorem,

$$\begin{aligned}
\int_\Omega \nabla\cdot\vec{v}dV &= \int_{\partial\Omega}\vec{v}\cdot d\vec{S}\\
&= \int_y^{y+\Delta y}\int_z^{z+\Delta z}\left(v_x(x+\Delta x,y',z')-v_x(x,y',z')\right)dz'dy'\\
&+ \int_z^{z+\Delta z}\int_x^{x+\Delta x}\left(v_y(x',y+\Delta y,z')-v_y(x',y,z')\right)dx'dz'\\
&+ \int_x^{x+\Delta x}\int_y^{y+\Delta y}\left(v_z(x',y',z+\Delta z)-v_z(x',y',z)\right)dy'dx'\\
&= \int_y^{y+\Delta y}\int_z^{z+\Delta z}\int_x^{x+\Delta x}\frac{\partial v_x}{\partial x}dx'dz'dy'\\
&+ \int_z^{z+\Delta z}\int_x^{x+\Delta x}\int_y^{y+\Delta y}\frac{\partial v_y}{\partial y}dy'dx'dz'\\
&+ \int_x^{x+\Delta x}\int_y^{y+\Delta y}\int_z^{z+\Delta z}\frac{\partial v_z}{\partial z}dz'dy'dx'\\
&= \int_x^{x+\Delta x}\int_y^{y+\Delta y}\int_z^{z+\Delta z}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)dz'dy'dx'\\
&= \int_\Omega\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)dV
\end{aligned}$$

The above equation has to hold at any region Ω. Therefore,

$$\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$$

The x component of the curl of $\vec{v}$ is

$$
\begin{aligned}
&(\nabla \cdot \vec{v}) \cdot \vec{e_x} = \lim_{\Delta y, \Delta z \to 0} \frac{\oint_C \vec{v} \cdot \vec{dr}}{\Delta y \Delta z} \\
&= \lim_{\Delta y, \Delta z \to 0} \frac{1}{\Delta y \Delta z} \left(\int_y^{y+\Delta y} v_y(x, y', z) dy' + \int_z^{z+\Delta z} v_z(x, y + \Delta y, z') dz' \right. \\
&\quad \left. + \int_{y+\Delta y}^{y} v_y(x, y', z + \Delta z) dy' + \int_{z+\Delta z}^{z} v_z(x, y, z') dz' \right) \\
&= \lim_{\Delta y, \Delta z \to 0} \frac{\int_z^{z+\Delta z} (v_z(x, y + \Delta y, z') - v_z(x, y, z')) dz' - \int_y^{y+\Delta y} (v_y(x, y', z + \Delta z) - v_y(x, y', z)) dy'}{\Delta y \Delta z} \\
&= \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}
\end{aligned}
$$

Similarly, we can compute the y component and z component, and we can obtain

$$
\nabla \times \vec{v} = \begin{pmatrix} \dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z} \\ \dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x} \\ \dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} \end{pmatrix}
$$

Theorem Green's Theorem.

$$
\oint_C P dx + Q dy = \int_\Gamma \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dS
$$

where $\Gamma \in \mathbb{R}^2$ which is closed by a contour C

Proof. Let $\vec{v}$ be defined as follows.

$$
\vec{v} = \begin{pmatrix} P \\ Q \\ 0 \end{pmatrix}
$$

Using the Stokes' theorem,

$$
\begin{aligned}
\oint_C \vec{v} \cdot \vec{dr} &= \oint_C Pdx + Qdy \\
&= \int_\Gamma (\nabla \times \vec{v}) \cdot \vec{ds} \\
&= \int_\Gamma \begin{pmatrix} -\dfrac{\partial Q}{\partial z} \\ \dfrac{\partial P}{\partial z} \\ \dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS \\
&= \int_\Gamma \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dS
\end{aligned}
$$

□