Cauchy–Riemann Equations

Theorem Cauchy–Riemann Equations.

If a complex-valued function $f : \mathbb{C} \to \mathbb{C}$ is analytic at $z(= x + iy)$, then the following equations hold. ($x \in \mathbb{R}$ and $y \in \mathbb{R}$)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ are real part and imaginary part of f. This means that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$

Proof. Since f is analytic at z, there exists $\alpha \in \mathbb{C}$ which satisfies,

$$f(z + \Delta z) - f(z) = \alpha \Delta z + o(\Delta z)$$

where o is a Landau's little-o notation. (This means that α does not change by how Δz approaches to zero.)

If we write this using u and v, we obtain

$$\begin{aligned} &(u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)) - (u(x, y) + iv(x, y)) \\ &\quad = \alpha\,(\Delta x + i\Delta y) + o\left(\sqrt{\Delta x^2 + \Delta y^2}\right) \end{aligned}$$

This equation holds even if we consider $\Delta y = 0$ and let Δx approach to zero. Hence,

$$(u(x + \Delta x, y) + iv(x + \Delta x, y)) - (u(x, y) + iv(x, y)) = \alpha \Delta x + o\,(\Delta x)$$

($o(|\Delta x|)$ is equivalent to $o(\Delta x)$)

Therefore,

$$\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} = \alpha + \frac{o\,(\Delta x)}{\Delta x}$$

As $\Delta x \to 0$,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \alpha$$

Similarly, we consider $\Delta x = 0$ and make Δy approach to zero.

$$(u(x, y + \Delta y) + iv(x, y + \Delta y)) - (u(x, y) + iv(x, y)) = i\alpha\Delta y + o\left(\Delta y\right)$$

Therefore,

$$-i\frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} = \alpha + \frac{o\left(\Delta y\right)}{\Delta y}$$

As $\Delta y \to 0$,

$$-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \alpha$$

By comparing α obtained by the above, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

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